Summary Picture of How to Prove Inv. Func. Th.

Wolog $F : B(\vec{0}, \varepsilon) \subset \mathbb{R}^n \to \mathbb{R}^n$ with $F(\vec{0}) = \vec{0}$, $dF(\vec{0}) = I$ and $\|DF - I\| \le \frac{1}{2}$ on $B(\vec{0}, \varepsilon)$.

Set $G(\vec{x}) = F(\vec{x}) - \vec{x}$. Then $\|DG\| \le \frac{1}{2}$ on $B(\vec{0}, \varepsilon)$. If $F(\vec{x}_1) = F(x_2)$ then

$$\| G(x_2) - G(x_1) \| = \| F(x_2) - x_2 - (F(x_1) - x_1) \| = \| x_1 - x_2 \|$$

But $\| G(x_2) - G(x_1) \| \le \frac{1}{2} \| x_2 - x_1 \| = \frac{1}{2} \| x_1 - x_2 \|$ because of M Value Estimate (using $\|DG\| \le \frac{1}{2}$)

These two are contradictory ($\| G(x_2) - G(x_1) \| = \| x_1 - x_2 \|$ and $\le \frac{1}{2} \| x_1 - x_2 \|$) unless $\| x_1 - x_2 \| = 0$ or $x_1 = x_2$.

Thus F is one-to-one (injective) on $B(\vec{0}, \varepsilon)$.

F is "onto" $B(\vec{0}, \varepsilon/2)$ by the contraction mapping argument on the other page, that is $F(B(\vec{0}, \varepsilon)) \supset B(\vec{0}, \varepsilon/2)$. This shows that F on $B(\vec{0}, \varepsilon)$ is actually an "open mapping" (at a point $\vec{x} \ne \vec{0}$, $\vec{x}$ in $B(\vec{0}, \varepsilon)$, we could do the same argument to show $F(B(\vec{0}, \varepsilon)) \supset$ some open ball around $F(\vec{x})$).

How to Prove the "Locally Onto" Part of the Inverse Function Theorem.

Suppose (WOLOG) $F(\vec{0})=\vec{0}$ $dF(\vec{0})=I$ and $\varepsilon>0$ $\ni$ F is defined and continuously differentiable on $B(\vec{0},\varepsilon)$ $(=\{v: \|v\|<\varepsilon\})$ and that

$\|DF|_v - I\| < \frac{1}{2}$, $\forall v\in B(\vec{0},\varepsilon)$.

Claim: $F(B(\vec{0},\varepsilon)) \supset B(\vec{0},\varepsilon/2)$.

Proof: For $y\in B(\vec{0},\varepsilon/2)$ given, set $H(\vec{x})=\vec{x}-F(\vec{x})+y$.

If $\vec{x}\in B(\vec{0},\varepsilon)$, then $\|\vec{x}-F(\vec{x})\|\le \frac{1}{2}\|\vec{x}\|$ by the MV Inequality. [Detail: $\|\vec{x}-F(\vec{x})\| = \|G(\vec{x})\|$ with $G=F-I$. $\|G(\vec{x})\| = \|G(\vec{x})-G(\vec{0})\| \le \frac{1}{2}\|\vec{x}-\vec{0}\| = \|\vec{x}\|$ since $G(\vec{0})=\vec{0}$ and $\|dG\|<\frac{1}{2}$ on $B(\vec{0},\varepsilon)$ by hypothesis]. So $H(\vec{x})\in B(\vec{0},\varepsilon)$ for all $\vec{x}\in B(\vec{0},\varepsilon)$ if $y\in B(\vec{0},\varepsilon/2)$. Also $\|dH\|\le\frac{1}{2}$ on $B(\vec{0},\varepsilon)$.

Set $x_1=H(\vec{0})=y$, $x_2=H(x_1)$, $x_3=H(x_2)$, ...

Then $\|x_2-x_1\| = \|H(x_1)-H(\vec{0})\| \le \frac{1}{2}\|x_1-0\| = \frac{1}{2}\|y\| < \varepsilon/4$ so $x_2\in B(\vec{0},\frac{\varepsilon}{2}+\frac{\varepsilon}{4})$

Similarly $\|x_3-x_2\|\le \frac{1}{2}\|x_2-x_1\| \le \frac{1}{2}(\frac{1}{2}\|y\|) = \frac{1}{4}\|y\| < \frac{\varepsilon}{8}$

so $x_3\in B(\vec{0},\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{8})$...

Inductively $x_n\in B(\vec{0},\varepsilon)$ and indeed $x_n\in B(\vec{0},2\|y\|)$ for all n. (So x_n continues to be defined for all n in particular.) Also $\{x_n\}$ is a Cauchy sequence.

So $x_0=\lim x_n$ exists and is in $\{v: \|v\|\le 2\|y\|\}\subset B(\vec{0},\varepsilon)$.

Clearly $H(x_0)=x_0$. So $\vec{x}_0-F(x_0)+y=\vec{x}_0$ or $F(\vec{x_0})=y$.

Thus $F(B(\vec{0},\varepsilon))\supset B(\vec{0},\varepsilon/2)$ as we claimed. $\square$

The Successive Approximations Idea (Essentially Equivalent Alternative to Contraction Mapping)

$F(\vec{0})=\vec{0}$ $\quad dF(\vec{0})=I$ $\quad \| dF|_p - I\| \le \frac{1}{2}$ $\quad p\in B(\vec{0},\varepsilon)$

(as before) $\quad y\in B(\vec{0},\varepsilon/2)$

Set $x_1 = y$ (first guess for something with $F(\vec{x})=y$)

$x_{n+1} = x_n + \underbrace{(-F(x_n)+y)}$ ← natural correction if $dF=I$

Now need: $F(x+h) = F(x)+h+E$ where $\|E\|\le\frac{1}{2}\|h\|$ (provided $x+h,\ x \in B(\vec{0},\varepsilon)$). This follows from

$\| F(x+h) - F(x) - h\| = \| F(x+h) - F(x) - ((x+h)-x)\|$

$= \| G(x+h) - G(x)\| \le \frac{1}{2}\|h\|$ where $G(\cdot) = F(\cdot)-(\cdot)$

as before.

So

$$F(x_{n+1}) = F(x_n + -F(x_n)+y) = F(x_n) + -F(x_n) + y + E = y+E$$

where $\|E\| \le \frac{1}{2}\|y-F(x_n)\|$. Thus

$$\| F(x_{n+1}) - y\| \le \frac{1}{2}\|y - F(x_n)\| \text{ so } \|F(x_n)-y\|\le 2^{-n+1}\|y\|$$

In particular $\|x_n\| \le \|\overset{x_1}{y}\| + \|x_2-x_1\| + \|x_3-x_2\| + \cdots + \|x_n - x_{n-1}\|$

$\le \|y\| + \frac{\|y\|}{2} + \frac{\|y\|}{4} + \cdots + \frac{\|y\|}{2^{n-1}}$ $\quad$ since this is $\|y-F(x_2)\|$

$\le 2\|y\|$ $\quad$ since $\|x_2 - x_1\| = \|y - F(x_1)\|$

$<\varepsilon$.

So sequence $\{x_n\}$ remains in $\{v : \|v\| \le 2\|y\|\} \subset B(\vec{0},\varepsilon)$.

The sequence is Cauchy and $x_0 = \lim x_n$ satisfies

$F(x_0)=y$ $\quad$ (since n large $\Rightarrow \|F(x_n)-y\|$ small).

Solving Functional Equations by Contraction Mapping: An Example ④

Let $f(x) = x + \frac{1}{4}x^2$. Then $|f' - 1| = |\frac{1}{2}x| < \frac{1}{2}$ on $(-1, 1)$. By our contraction mapping principle, we should be able to solve, for $y \in (-\frac{1}{2}, \frac{1}{2})$, $f(x_0) = y$ by setting $H(x) = x - f(x) + y = -\frac{1}{4}x^2 + y$ and letting $x_0 = \lim H(H \ldots H(0))$.

Then $H(x_0) = x_0$ so $x_0 - f(x_0) + y = x_0$ or $f(x_0) = y$.

Let $x_1 = H(0) = y$, $x_2 = H(x_1)$, $x_3 = H(x_2)$, etc.

Then

$$x_1 = y$$

$$x_2 = -\frac{1}{4}y^2 + y$$

$$x_3 = -\frac{1}{4}(y - \frac{1}{4}y^2)^2 + y = y - \frac{1}{4}y^2 + \frac{1}{8}y^3 - \frac{1}{64}y^4$$

$$x_4 = -\frac{1}{4}(y - \frac{1}{4}y^2 + \frac{1}{8}y^3 - \frac{1}{64}y^4)^2 + y$$

$$= y - \frac{1}{4}y^2 + \frac{1}{4}(+\frac{2}{4}y^3) \ldots$$

$$x_5 = y - \frac{1}{4}(y - \frac{1}{4}y^2 + \frac{1}{8}y^3 \ldots)^2 = y - \frac{1}{4}y^2 + \frac{1}{8}y^3 \ldots \text{etc.}$$

$\vdots$

Now we can solve $x + \frac{1}{4}x^2 = y$ explicitly by the quadratic formula $x^2 + 4x - 4y = 0$ has solutions

$$\frac{1}{2}(-4 \pm \sqrt{16 + 16y}) = -2 \pm \sqrt{4 + 4y} = -2 + 2((1+y)^{\frac{1}{2}})$$

$$= -2 + 2(1 + \frac{y}{2} - \frac{1}{8}y^2 + \frac{3}{48}y^3 \ldots) = y - \frac{1}{4}y^2 + \frac{\cancel{3}\,1}{\cancel{48}\,16}2y^3 \ldots$$

$$= y - \frac{1}{4}y^2 + \frac{1}{8}y^3 \ldots$$

So you can see the Contraction Mapping sequence turning into the power series expansion derived from the Quadratic Formula & Taylor series of $\sqrt{1+y}$.

(Do some more terms if you feel like it!)